

ON THE EXISTENCE OF ORTHOGONAL POLYNOMIALS FOR OSCILLATORY WEIGHTS ON A BOUNDED INTERVAL

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ABSTRACT. It is shown that the orthogonal polynomials, corresponding to the oscillatory weight $e^{i\omega x}$, exists if ω is a transcendental number and $\tan \omega/\omega \in \mathbb{Q}$. Also, it is proved that such orthogonal polynomials exist for almost every $\omega > 0$, and the roots are all simple if $\omega > 0$ is either small enough or large enough.

Keyword. orthogonal polynomial; oscillatory weight; Gaussian quadrature rule

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1. INTRODUCTION

We consider the problem of existence of orthogonal polynomials and Gaussian quadrature rules (in the standard form) for the following inner product:

$$(f, g)_\omega = \int_{-1}^1 f(x)g(x)e^{i\omega x} dx, \quad (1)$$

with $\omega > 0$. More precisely, we seek a monic polynomial p_n^ω of a given degree n such that

$$\int_{-1}^1 p_n^\omega(x)x^j e^{i\omega x} dx = 0, \quad j = 0, 1, \dots, n-1. \quad (2)$$

The following results on the existence of p_n^ω are due to [1]:

Proposition 1: p_1^ω exists for any ω except when ω is a multiple of π ;

Proposition 2: p_2^ω exists for all ω ;

Conjecture 1: p_n^ω with n even exists for all ω ;

Conjecture 2: p_n^ω with n odd does not exists for some ω .

In this paper, we give a sufficient condition on ω for which p_n^ω exists for all n . According to Conjecture 1, this condition is not necessary. We show that p_n^ω exists for almost every $\omega > 0$. If the existence of p_n^ω is assumed, it is shown that all of its roots are simple when $\omega > 0$ is either small enough or large enough.

Throughout the paper, we frequently suppress the dependence of objects on ω for simplification in notations.

2. ORTHOGONAL POLYNOMIALS

A necessary and sufficient condition for existence of the orthogonal polynomial p_n^ω is that the Hankel determinant

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} \quad (3)$$

does not vanish. The moment $\mu_k := \int_{-1}^1 x^k e^{i\omega x} dx$ is defined recursively (see [1]):

$$\mu_0 = \frac{2 \sin \omega}{\omega}, \quad (4a)$$

$$\mu_k = \frac{1}{i\omega} (e^{i\omega} - (-1)^k e^{-i\omega}) - \frac{k}{i\omega} \mu_{k-1}, \quad k \geq 1. \quad (4b)$$

It is easy to show that

$$\mu_k = \frac{(-1)^k k!}{(i\omega)^k} \sum_{\nu=0}^k \frac{(-i\omega)^\nu s_\nu}{\nu!}, \quad (5)$$

where

$$s_\nu := \frac{1}{i\omega} (e^{i\omega} - (-1)^\nu e^{-i\omega}) = \begin{cases} \frac{2 \sin \omega}{\omega}, & \text{for } \nu \text{ even,} \\ \frac{2 \cos \omega}{i\omega}, & \text{for } \nu \text{ odd.} \end{cases}$$

Then we can expand (5) into

$$\mu_k = \frac{2(-1)^{k+1}k!}{(i\omega)^k} \left(\cos \omega \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^k \frac{(-i\omega)^{\nu-1}}{\nu!} - \frac{\sin \omega}{\omega} \left(1 + \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^k \frac{(-i\omega)^\nu}{\nu!} \right) \right). \quad (6)$$

Now consider the matrix corresponding to the Hankel determinant Δ_n . If we take from the r th row the factor $\left(\frac{-1}{i\omega}\right)^{r-1}$, and from the s th column the factor $\left(\frac{-1}{i\omega}\right)^{s-1}$, then we arrive at a new Hankel determinant $\tilde{\Delta}_n$ with the moments

$$\tilde{\mu}_k := -2k! \left(\cos \omega \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^k \frac{(-i\omega)^{\nu-1}}{\nu!} - \frac{\sin \omega}{\omega} \left(1 + \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^k \frac{(-i\omega)^\nu}{\nu!} \right) \right). \quad (7)$$

The relation between Δ_n and $\tilde{\Delta}_n$ is then

$$\Delta_n = \left(\frac{1}{i\omega}\right)^{n(n-1)} \tilde{\Delta}_n.$$

Thus, $\tilde{\Delta}_n \neq 0$ if and only if $\Delta_n \neq 0$. If ω is such that each $\tilde{\mu}_k$ is a polynomial in $i\omega$ with rational coefficients, then $\tilde{\Delta}_n$ is a polynomial in $i\omega$ with rational coefficients. As the proof of Theorem 2.3 in [2], we employ the fact that transcendental numbers can not be zeros of a polynomial with rational coefficients. Then we seek a set S of transcendental ω , for which $\tilde{\mu}_k$ is a polynomial in $i\omega$ with rational coefficients. Clearly, any multiplier of π falls in S .

If $\omega \in S$, then $\cos \omega \neq 0$. Then the moments can be rewritten as

$$\mu_k = \frac{2(-1)^{k+1}k! \cos \omega}{(i\omega)^k} \left(\sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^k \frac{(-i\omega)^{\nu-1}}{\nu!} - \frac{\tan \omega}{\omega} \left(1 + \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^k \frac{(-i\omega)^\nu}{\nu!} \right) \right). \quad (8)$$

Again using the above idea, it is enough to determine $\omega > 0$ not belonging to \mathbb{Q} (the field of rational numbers) for which

$$\hat{\mu}_k := -2k! \left(\sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^k \frac{(-i\omega)^{\nu-1}}{\nu!} - \frac{\tan \omega}{\omega} \left(1 + \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^k \frac{(-i\omega)^\nu}{\nu!} \right) \right) \quad (9)$$

is a polynomial in $i\omega$. Thus, the problem is to find transcendental numbers $\omega > 0$ not belonging to $\{m\pi : m = 1, 2, \dots\}$, such that $\tan \omega / \omega \in \mathbb{Q}$.

Transcendental numbers can be zeros of a polynomial with rational coefficients if and only if the polynomial is identically zero. Thus it is enough to show that $\tilde{\Delta}_n, \hat{\Delta}_n$, as functions of $i\omega$, are not identically zero for $n > 1$. This can be shown by a discussion similar to that carried out in the proof of Theorem 2.3 in [2]. Thus we have the following result.

Proposition 2.1. *For any transcendental $\omega > 0$ with $\tan \omega / \omega \in \mathbb{Q}$, the orthogonal polynomial p_n^ω exists.*

Remark 2.1. The converse is not necessarily true. There are examples of ω with $\tan \omega / \omega \notin \mathbb{Q}$ while $\Delta_n \neq 0$, i.e., the orthogonal polynomial p_n^ω exists. For example, p_2^ω exists for any $\omega > 0$ [1].

The set S determined in Proposition 2.1 is at most countable due to countability of \mathbb{Q} . However, our numerical experiences show that p_n^ω exists for almost every $\omega > 0$. In the following, we establish this result.

Theorem 2.2. *p_n^ω exists for almost every $\omega > 0$.*

Proof. By induction on the index k , we can show from (4) that the moments μ_k , as functions of ω , are analytic in D , an arbitrary connected neighborhood of the semi-axis $\omega > 0$. The same result holds then for the Hankel determinant $\Delta_n = \Delta_n(\omega)$. Since zeros of any analytic function (if it is not identically zero) are isolated, it is enough to show that $\Delta_n(\omega)$ is not identically zero in D . Since Δ_n is analytic and then continuous, it is enough to show that $\Delta_n(0) \neq 0$; and this can be done similar to the proof of Theorem 2.3 in [2]. \square

3. GAUSSIAN QUADRATURE RULES

Since the weight function in (1) is not positive, we can not readily claim that the roots of p_n^ω (if exists) are all simple. If p_n^ω have some multiple zeros, then the n -point Gaussian quadrature rule can be written in the following form:

$$G_n(g) = \sum_{\nu=1}^n \sum_{k=0}^{m_\nu-1} w_{\nu,k} f^{(k)}(x_\nu),$$

where m_ν is the multiplicity of the node x_ν , and the weights $w_{\nu,k}$ are such that the rule is exact if f is replaced by a polynomial of degree at most $2n - 1$. Here in the notations, we suppressed the dependence of the nodes and the weights on n . This rule, however, is rarely of practical interest since determining the multiplicities of the nodes is not an easy task. Our numerical experiences show that the roots of p_n^ω (if exists) are all simple.

This result can be established if we assume the existence of p_n^ω for all $\omega > 0$. According to Conjecture 2, this result most probably holds for n even. From our numerical experiences, the same result can be drawn too. We have computed the absolute values of the Hankel determinant for $n = 2, 4, 6$; for each n , the graph has been drawn for some increasing ω (see Figure 1). As it is seen, the graphs never cut the horizontal axis, i.e., the Hankel determinants never vanish.

Lemma 3.1. *For a given integer $n > 0$, assume that the orthogonal polynomial $p_n^\omega(x)$ exists for all $\omega > 0$. Then all coefficients of $p_n^\omega(x)$ as functions of ω are continuous.*

Proof. If $p_n^\omega(x) = x^n + \sum_{k=0}^{n-1} a_k(\omega)x^k$, then the coefficients $a_0(\omega), \dots, a_{n-1}(\omega)$ satisfy the linear system

$$[v_0(\omega), \dots, v_{n-1}(\omega)] u_n(\omega) + v_n(\omega) = 0, \quad (10)$$

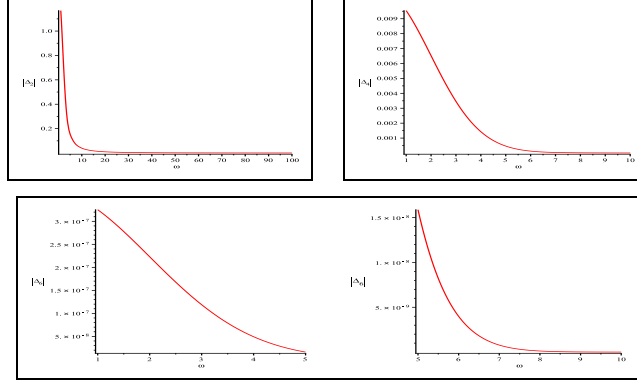


FIGURE 1. Absolute values of the Hankel determinants Δ_n for $n=2,4,6$.

where

$$v_k(\omega) = [\mu_k, \mu_{k+1}, \dots, \mu_{k+n-1}]^T, \quad u_n(\omega) = [a_0(\omega), \dots, a_{n-1}(\omega)]^T.$$

Then

$$u_n(\omega) = -\frac{1}{\Delta_n} [V_n(\omega)]^T v_n(\omega), \quad (11)$$

where $V_n(\omega)$ is the cofactor matrix of $[v_0(\omega), \dots, v_{n-1}(\omega)]$. All entries of the matrix $V_n(\omega)$ are continuous with respect to ω due to the continuity of the moments μ_k , the entries of $[v_0(\omega), \dots, v_{n-1}(\omega)]$. Since the denominator Δ_n does not vanishes for any $\omega > 0$, the result follows from (11). \square

Theorem 3.2. *For a given integer $n > 0$, assume that the orthogonal polynomial $p_n^\omega(x)$ exists for all $\omega > 0$. If $\omega > 0$ is small enough or large enough, then all of the roots of the orthogonal polynomial $p_n^\omega(x)$ are simple.*

Proof. It is well-known that the roots of a polynomial vary continuously as the coefficients of the polynomial change continuously. Thus, Lemma 3.1 implies that the trajectories of the roots of $p_n^\omega(x)$, as $\omega > 0$ increases, are all continuous. Since the roots corresponding to $\omega = 0$ as well as $\omega \rightarrow \infty$ are all distinct [1], then the result follows. \square

4. CONCLUDING REMARKS

We have shown that the orthogonal polynomial p_n^ω , corresponding to the oscillatory weight $e^{i\omega x}$, exists if ω is a transcendental number and $\tan \omega/\omega \in \mathbb{Q}$. The set of such ω is nonempty since it contains the multipliers of π . Determining other members is not an easy task, so the main problem is still unsolved: For which values of ω does p_n^ω exist? We have also shown that p_n^ω exist for almost every ω .

In order to arrive at an n -point Gaussian quadrature rule of standard form, it is necessary that all the roots of p_n^ω (if exists) to be simple. The simplicity of the roots of p_n^ω is established only when $\omega > 0$ is small enough or when it is large enough. The problem is unsolved for an arbitrary $\omega > 0$. We believe that the more properties of p_n^ω one knows, the higher chance he has to solve the problem. For instance, the symmetricity of p_n^ω (cf. [1]) implies that the coefficients of $p_n^\omega(z)$ (starting from 1, the coefficient of z^n) are real and pure imaginary, alternatively. Also from the three-term recurrence relation,

$$p_k^\omega(z) = (z - \alpha_{k-1})p_{k-1}^\omega(z) - \beta_{k-1}p_{k-2}^\omega(z), \quad (12)$$

and Theorem 3.3 of [1], it is easy to show that α_k and α'_k are pure imaginary numbers; β_k and β'_k are real. Here the prime sign indicates the derivative with respect to ω .

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